# TOPOLOGICAL ANALYSIS
# AND SYMMETRY GROUPS
# FOR CHAOTIC DYNAMICAL SYSTEMS

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## I. Introduction

A dynamical system is a system coming from physics, biology, chemistry or mathematics which is described in terms of ordinary differential equations, that is, by variables evolving versus time. In such a case, the evolutions can be predicted – at least for a small interval of time – from initial conditions. As introduced by Poincaré, a nonlinear system can rarely be integrated. This means that no analytical solution can be found. It is therefore useful to investigate the dynamics in the phase space spanned by variables involved in the differential equations. The deterministic character is thus identified by a structure in the phase space. Contrary to this, a stochastic dynamics would present an evolution without any structure in the phase space. In this paper, the case of deterministic dynamical systems with symmetry properties is investigated. Attractors with symmetry may have a topological structure quite complicated. Their analysis thus requires a special procedure taking into account the symmetry. Such a procedure is based on the fact that dynamics – as physics – is invariant under symmetries. Moreover, it explains why the Lorenz system which has a rotation symmetry has a Fourier spectrum that depends on the chosen variable.

## II. Modding out or introducing symmetry properties

According to a procedure introduced by Miranda & Stone [1], it is possible to modd out symmetry properties by using a coordinate transformation. Let us start with the example of the Lorenz system [2] which is invariant under a rotation symmetry by p around the $z$-axis defined by the coordinate transformation $(x,y,z)\to(-x,-y,z)$. This means that the attractor is globally invariant under this coordinate transformation. This symmetry is defined by the $3\times 3$ matrix

$$\Gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is an order 2 symmetry since $\Gamma^2 = \mathrm{Id}$ where *Id* designates the identity. When a system has an order-2 rotation symmetry around the *z*-axis, a possible coordinate transformation to modd out the symmetry is

$$\Psi_2 = \left| \begin{array}{l} u = \Re(x+iy)^2 = x^2 - y^2 \\ v = \Im(x+iy)^2 = 2xy \\ w = z \end{array} \right.$$

Thus the two wings Lorenz attractor becomes an attractor without any symmetry property, that is, a single wing attractor.

When a system has no symmetry, it is possible to introduce a rotation symmetry by inverting the coordinate transformation $\Psi_2$. For instance, a rotation symmetry can be introduced in the Rössler system by applying $\Psi_2^{-1}$ [3]. When an order-*n* rotation is wanted, it is sufficient to invert the map

$$\Psi_n = \left| \begin{array}{l} u = \Re(x+iy)^n \\ v = \Im(x+iy)^n \\ w = z \end{array} \right. .$$

It is thus possible to have a Rössler attractor with *n* wings. The case of the double-cover of the Rössler attractor is shown in Fig. 1.

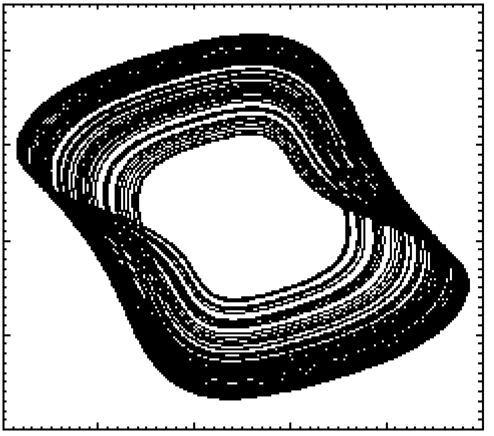
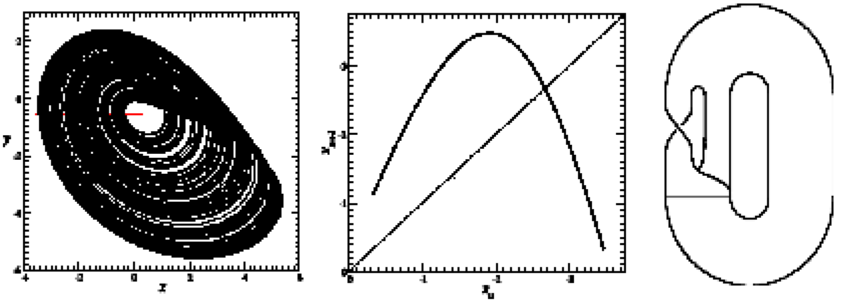


Fig. 1: Using the coordinate transformation $\Psi_2^{-1}$, a rotation symmetry by $\pi$ around the z-axis is introduced into the Rössler attractor: two foldings, one being symmetric from the other, are obviously shown.

## III. Topological characterization

A chaotic attractor is organized in the neighbourhood of a set of unstable periodic orbits. These orbits are bones for attractors. In topology, only the relative organization of these periodic orbits is taken into account. These orbits can be labelled by using a symbolic dynamics. For the spiral Rössler attractor, a first-return map to a Poincaré section is characterized by a parabola (two monotonic branches). The increasing branch is associated with symbol “0” and the decreasing branch with symbol “1”. The period-2 orbit with one periodic point located in the increasing branch and one in the decreasing branch is thus encoded by (10). It is thus possible to identify each orbit with a specific sequence.

Once periodic orbits identified, some topological invariants are computed. A topological invariant is a number which is not changed under an isotopy, that is, when the topology is preserved. A topological invariant often used is the linking number. It can be estimated by the half sum of the orientated crossings between two periodic orbit counted in a plane projection. Crossings are orientated according to the third coordinates of the two orbits [4]. The information provided by linking numbers can be synthesized in a branched manifold, also named template or knot-holder [5]. A template is validated when the linking numbers it predicts correspond to those estimated in plane projections of orbits extracted from the attractor studied. A brief introduction to topological characterization can be found in Ref. [6].



(a) (b) (c)

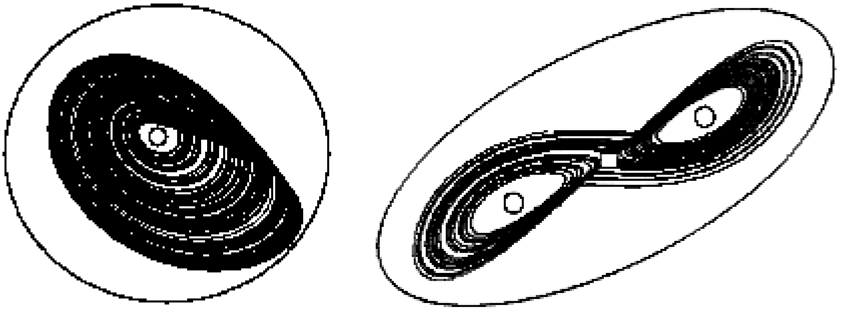
Fig. 2: Spiral Rössler attractor (a), a first-return map to a Poincaré section of it (b) and a template (c) synthesizing its topological properties.

When the system has no symmetry, the Poincaré section is simply computed by using a single half plane containing the inner fixed point. For instance, a Poincaré section for the spiral Rössler attractor can be defined by the red line as shown in Fig. 2a. But when the attractor presents an $n$-order symmetry, the Poincaré section is the union of $n$ components, one for each wing [7,8]. In fact, an attractor with an order-$n$ symmetry can be considered as the union of $n$ copies of a fundamental domain. There is one component of the Poincaré section associated with each copy. The number of components required to properly build a Poincaré section is easily obtained using bounding tori.

Periodic orbits are the basic elements of the attractor sketched by a branched manifold. At an upper level, the attractor can be embedded in a closed semi-permeable surface. In a generic way, this surface defines a torus whose genus corresponds to the number of fixed points surrounded by the flow [9]. The Rössler attractor is bounded by a genus-1 torus (Fig. 3a) and the Lorenz system by a genus-3 torus (Fig. 3b). To each hole is associated a focus (a circle in Fig. 3) or a saddle (a square as in Fig. 3b). The number of components for a Poincaré section is given as follows. Let $N_F$ be the number of foci surrounded by the flow, $N_S$ the number of saddles surrounded by the flow and $N_p$ the number of components of the Poincaré section. Then $N_p$ is given by

$$N_P = \begin{cases} 1 & N_F + N_S = 1 \\ 2 \quad \text{if} & N_F + N_S = 3 \\ N_F + N_S - 1 & N_F + N_S > 3 \end{cases}$$

This formula holds for any kind of attractor, with or without symmetry.



(a) Genus-1 torus (b) Genus-3 torus

Fig. 3: Bounding tori associated with the spiral Rössler attractor (a) and the Lorenz attractor (b).

There is one special case when a focus is degenerated. This can happen when the rotation axis contains the inner focus. A geed example of such situation is the two-fold cover of the spiral Rössler attractor. The resulting attractor (Fig. 1) is bounded by a genus-1 torus but the single hole is degenerated three times. This hole is in fact associated with two foci and one saddle. This feature is observed when the trajectory does not visit the neighbourhoods between the saddle and the foci, there is thus a single hole – and not three – in the middle of the attractor. In order to obtain the right number of components, we should consider that there is, in fact, two foci and one saddle. A two-components Poincaré section must therefore be taken into account [3,8].

## IV. Image system to simplify the analysis

Using the procedure described in Section II, it is possible to remove the symmetry from the Lorenz system to obtain a so-called image system. An attractor bounded by a genus-1 torus is thus obtained [3]. A Poincaré section for this attractor is made of a single component and the so-called «Lorenz map» is easily obtained. This map can be obtained from the original Lorenz attractor by using the first return to maxima of the $z$-variable as Lorenz did in his original paper [9]. This procedure is equivalent to compute a Poincaré section made of two components defined by $z=R-1$ for instance and to identify them by taking the absolute value of the $x$-coordinate of each intersection with these two components [10].

The Lorenz system also presents a particular feature. Two different types of Fourier spectra are obtained from variables of the Lorenz system. From variables $x$ and $y$, the spectra do not present a main peak as from the $z$-variable [10]. When a Fourier spectrum is computed from any variable from the image system of the Lorenz system, a spectrum equivalent to those obtained from variable $z$ is observed. Thus, there is a frequency characteristic of revolutions around the focus in the image system or, equivalently, revolutions around foci in the original system. In fact, switches from one wing to the other blurr the time coherence of the Lorenz system and the main peak is diluted in the broad band.

## V. Some examples with groups $V_4$ and $S_4$

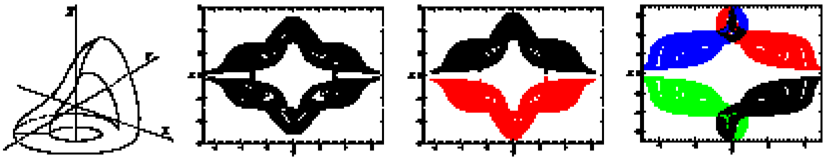
Two examples with more exotic symmetries are now discussed. The first examples is equivariant under the vier-gruppe (introduced by Felix Klein) associated with the four operators

$$\underset{\mathrm{R}_x(\pi)}{\sigma_1 = \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}} \quad \underset{\mathrm{R}_y(\pi)}{\sigma_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix}} \quad \underset{\mathrm{R}_z(\pi)}{\sigma_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}} \quad \underset{\mathrm{Id}}{\mathrm{Id} = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}}.$$

A $V_4$-cover of the Rössler system can be obtained using the coordinate transformation [11]

$$\Psi_{V_4} = \left| \begin{array}{l} u = \dfrac{x^2 - y^2}{2} \\ v = \dfrac{x^2 + y^2 - 2z^2}{2} \\ w = xyz \end{array} \right. .$$

The cover thus obtained corresponds to locations of the rotations axis as shown in Fig. 4a. The attractor thus obtained is shown in Fig. 4b. Depending on the location of the z-axis, one, two or four attractors can co-exist in the phase space.



(a) Locations of rotation axis (b) Attractor(s) in the phase space

Fig. 4: Attractors solution to the $V_4$-cover of the Rössler system.

A second example is invariant under the group $S_4$ made of the four operators $\{d, s_4, s_4^2, s_4^3\}$ where

$$s_4 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

results from a reflexion combinated with a rotation by p/2 around the *z*-axis. A possible system with this symmetry property is the slightly modified Leipnik-Newton system [11]

$$\begin{cases} \dot{x} = -ax + y + byz \\ \dot{y} = -x - ay + bxz \\ \dot{z} = -bxy + cz \end{cases}$$

With parameter values *a*=0.6, *b*=5 and *c*=0.1428, there are four co-existing attractors (Fig. 5). Each attractor is a copy of the others through the action of one of the operators of the group $S_4$.

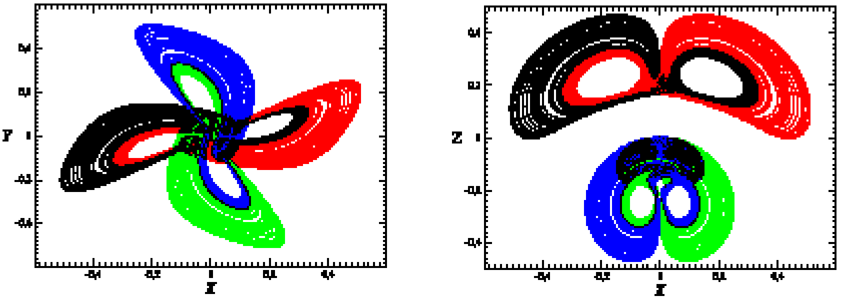



Fig. 5: Four co-existing attractors solution to the modified Leipnik-Newton.

## VI. Conclusion

Identifying symmetry groups helps to classify dynamical systems and to analyse them. It is also possible to introduce some symmetry (covering) and to modd out symmetry (image system). Some examples have been discussed with different types of symmetries. These symmetries can be used to investigate in an appropriate way some data from the real world. For instance, the dynamics underlying the sunspot numbers was characterized by introducing the symmetry related to the magnetic field reversals [12,13].

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