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OTRAS COMUNICACIONES CIENTÍFICAS

INTERNAL SYMMETRIES IN HOMOGENEOUS AND ISOTROPIC UNIVERSES

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Abstract

We find the group of symmetry transformations under which the Einstein-Klein-Gordon (EKG) equations, for the spatially flat Friedmann-Robertson-Walker (FRW) universe, are form invariant. They relate the energy density and the pressure of the fluid with the expansion rate. We show that inflation can be obtained from non-accelerated scenarios by a symmetry transformation. Also, the concept of form-invariance can be used for generating phantom cosmologies involving linear transformations between the kinetic energy and the potential of the scalar field and, generalizing the well-known $a \rightarrow a^{-1}$ duality. They transform solutions of the EKG equations which satisfy the weak energy condition into others which violate it, preserving the energy density of the field.

Resumen

Encontramos el grupo de transformaciones de simetría que dejan invariante la forma de las ecuaciones de Einstein-Klein-Gordon (EKG) para un universo de Friedmann-Robertson-Walker (FRW). Estas relacionan la densidad de energía y la presión del fluido con el factor de expansión. Mostramos que la inflación puede obtenerse a partir de escenarios no acelerados mediante una transformación de simetría. Aplicamos el concepto de invariancia de forma para generar cosmologías superaceleradas usando transformaciones lineales del potencial y la energía cinética del campo escalar, generalizando la bien conocida dualidad $a \rightarrow a^{-1}$. Estas transforman soluciones de las ecuaciones de EKG que satisfacen las condiciones de energía débil en otras que la violan, conservando la densidad de energía del campo.

I. Introduction

There exist several interesting physical problems where the Einstein field equations for homogeneous, isotropic and spatially flat cosmological models [1]-[4] and Bianchi I-type metric [5] with a variety of matter sources, can be linearized and solved by writing them in invariant form [6]. In all these cases explicit uses of the non-local transformation group has been made. On the other hand, recent obtained astrophysical information ranging from high redshift surveys of supernovae to Wilkinson Microwave Anisotropy Probe (WMAP) observations indicates that a considerable amount of the total energy of our universe might correspond to some dark energy which would be currently producing a late accelerated stage thanks to a large and negative pressure. In this paper we focus our attention in the group of transformations leading to accelerate expansion scenarios. In many of these scenarios the effective potential energy density of a scalar field is responsible for an epoch of accelerated inflationary expansion [7]. In particular the so-called assisted inflation [8] will be investigated in detail. It was introduced in the context of standard FRW cosmology where cumulative effects of multiple scalar fields with an exponential potential give rise to inflation, provided that these fields interact only through the geometry. This kind of potential arises in various higher dimensional supergravity [9] and superstring [10] models [11]-[14]. We will see that, the cooperative effects of adding energy density into the Friedmann equation lead to inflation. So, the configurations of one and several scalar fields are related by a simple symmetry transformation of the respective EKG equations. In this sense they could be considered as equivalent cosmological models. Assisted inflation has been mainly studied in the case of power-law solutions ($a \alpha t^p$) for the spatially flat FRW cosmology, that can be shown [8] to be its late-time attractor. Here, we will investigate inflation without using any particular solution appealing to the symmetry group of the EKG equations. Also, from the point of view of symmetry transformations it will be interesting to investigate superaccelerated scenarios driven by exotic fluids that violate the weak energy condition [15]. These models were dubbed phantom cosmologies [16], and their study represents a currently active area of research in theoretical cosmology. Finally, we call the attention that the symmetry transformations preserving the form of the EKG equations may introduce an alternative concept of equivalence between different physical problems. We could say that several cosmological models are equivalent when the corresponding dynamical equations are form invariant under the action of that group. Thus, it turns out to be of great interest to investigate the consequences of this group from the physical and mathematical point of view. The paper is organized as follows; in section II we introduce the cosmological symmetry group of the EKG equations in FRW spacetime. In section III we investigate the consequences of this group on the assisted inflation in FRW spacetime. In section IV we examine the relation between duality and phantom cosmology. Finally in section V we will draw our main conclusions.

II. The form invariance symmetry in flat FRW spacetimes

The Einstein equations for a flat FRW cosmological model with scale factor a,

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2], \qquad (1)$$

and filled with a perfect fluid with energy density ρ and pressure p, are

$$3H^2 = \rho, \qquad (2)$$

$$\dot{\rho} + 3H(\rho + p) = 0, \qquad (3)$$

where $H = \dot{a}/a$ is the expansion rate. For a different perfect fluid with energy density $\bar{\rho}$ and pressure $\bar{p}$ the above equations become

$$3\bar{H}^2 = \bar{\rho}, \qquad (4)$$

$$\dot{\bar{\rho}} + 3\bar{H}(\bar{\rho} + \bar{p}) = 0. \qquad (5)$$

A form-invariance transformation is a prescription relating the quantities a, H, ρ, p of our original model with the quantities $\bar{a}$, $\bar{H}$, $\bar{\rho}$, $\bar{p}$ of the transformed model which preserves the form of the Einstein equations in flat FRW cosmology. The symmetry transformations linking the system of equations (4)-(5) with (2)-(3) is given by [17]

$$\bar{\rho} = \bar{\rho}(\rho), \qquad (6)$$

$$\bar{H} = \left(\frac{\bar{\rho}}{\rho}\right)^{1/2} H, \qquad (7)$$

$$\bar{\rho} + \bar{p} = \left(\frac{\bar{\rho}}{\rho}\right)^{1/2} \frac{d\bar{\rho}}{d\rho} (\rho + p), \qquad (8)$$

where $\bar{\rho} = \bar{\rho}(\rho)$ is an invertible function. Hence the FRW equations for a perfect fluid have a form invariance symmetry. The symmetry transformations (6)-(8) define a "cosmological symmetry group" generated by functions $\bar{\rho}(\rho)$. It can be used to solve the FRW equations and get accelerated expansion scenarios, as it will be seen in the following sections.

Now, we investigate the transformation properties of the relevant physical parameters. For instance, in the case of considering perfect fluids with equations of state $p = (\gamma - 1)\rho$ and $\bar{p} = (\bar{\gamma} - 1)\bar{\rho}$ respectively, the barotropic indices γ and $\bar{\gamma}$ transform as

$$\bar{\gamma} = \left(\frac{\rho}{\bar{\rho}}\right)^{3/2} \frac{d\bar{\rho}}{d\rho} \gamma \qquad (9)$$

under the symmetry transformations (6)-(8). On the hand, the expansion of the Universe is completely described by the scale factor, Hubble parameter and the deceleration parameter. In these models $q(t) = -H^{-2}\,\ddot{a}/a$, transforms as

$$\bar{q} + 1 = \left(\frac{\rho}{\bar{\rho}}\right)^{3/2} \frac{d\bar{\rho}}{d\rho} (q+1) \quad (10)$$

under the symmetry transformations (6)-(8). Besides, using (9) and (10) we get a form invariant relation $(\bar{q}+1)/\bar{\gamma}$ = (q + 1)/ γ between the deceleration parameter and the barotropic index.

Inflationary solutions occur when $\ddot{a} > 0$; this means that the expansion is dominated by a gravitationally repulsive stress that violates the strong energy condition, so that $\rho + 3p < 0$. Imposing this condition on (10) we obtain $d\bar{\rho}^{(-1/2)}/d\rho^{(-1/2)} < 1/(q+1)$, which for a non-accelerated cosmological model with $q \approx$ const > 0, gives $\bar{\rho} > (q+1)^2\rho$. Such model, with $\bar{q} < 0$, is accelerated. In the particular case which $\bar{\gamma}$ is constant, from (9), we get

$$a^{3\bar{\gamma}/2} = -\frac{\bar{\gamma}{\rho_0}^{1/2}}{2}\int \frac{d\rho}{\gamma\rho^{3/2}} \quad (11)$$

where ρ_0 is a constant. It shows that a cosmological model containing a perfect fluid with $\gamma = \gamma(\rho)$, can be mapped into one with constant adiabatic index, permitting to solve a complicated FRW model in term of an easier one. This can be seen as a way to obtain new exact solutions of the Einstein field equations taking profit of their form invariance symmetries.

We have seen that the form-invariance transformations require a relation between the energy densities of the two fluids. Thus, according to that, we may consider the linear transformation between H, ρ and p generated by

$$\bar{\rho} = n^2 \rho \quad (12)$$

where n is a real number. Even though the latter a simple choice, it has got much in store. Finally, for the linear relation (12), the transformation rule (6)-(8) between the physical quantities gives

$$\bar{H} = n H, \quad (13)$$

$$\bar{a} = a^n, \quad (14)$$

$$\bar{p} = -n^2 \rho + n(p + \rho), \quad (15)$$

where, Eq. (14) was obtained by integrating Eq. (13). Actually we are dealing with a linear transformation acting on the variables (ρ, H, p), which arises because of the particular choice (12). Also, the transformed barotropic index and deceleration parameter will read

$$\bar{\gamma} = \frac{\gamma}{n}, \quad \bar{q} = -1 + \frac{q+1}{n}. \quad (16)$$

There will be inflation ($\bar{q} < 0$) for any $n > (q + 1)$ if $n > 0$ or for any n $< (q + 1)$ if $n < 0$. The first case was studied in full detail in [17], whereas the second one was partially investigated in [18]. The insight gained in this section will be used in the remainder with the purpose of investigating assisted inflation, duality and phantom cosmologies.

III. Assisted inflation

The problem of a homogeneous scalar fields, ϕ, driven by an exponential potential $V(\phi) = V_0 e^{-k\phi}$ minimally coupled to gravity in the flat FRW spacetime is formulated by the system of equations

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V, \quad (17)$$

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \quad (18)$$

where V_0 and k are constants. Potentials of this type are of interest because they may be considered an approximation to more complex potentials. In fact, in higher dimensional superstring theories, the scalar field is like one of the matter fields that contribute to the action and effective potential of the theory. There is another important reason behind choosing exponential potentials, because there exists a late-time attractor solution for all the participating fields [8]. In this kind of models the EKG equations have power-law solutions

$$a \propto t^{2/k^2} \qquad (19)$$

so that they inflate at all times when $k^2 < 2$.

Let us now assume that n homogeneous scalar fields ϕ_I do not interact directly, but are driven by a sum of n exponential potentials $V_i = V_{0i} e^{-k_i \phi_i}$. From now on we consider a simplified problem in which the constants k_i and V_{oi} satisfy $k_1 = k_2 = \cdot \cdot \cdot = k_n = k$ and $V_{o1} = V_{o2} = \cdot \cdot \cdot = V_{0n} = V_0$. As the asymptotic evolution of all scalar fields tend to a common limit, the special scalar field configuration, in which all scalar fields are equal is a late-time attractor. We may take $\phi_1 = \phi_2 = \cdot \cdot \cdot = \phi_n$, so that $V_1 = V_2 = \cdot \cdot \cdot = V_n \equiv V_0 e^{-k\phi}$ and equations (17)-(18) become

$$3H^2 = n\left[\frac{1}{2}\,\dot{\phi}^2 + V\right], \qquad (20)$$

$$\ddot{\phi} + 3H\dot{\phi} - kV = 0. \qquad (21)$$

They have power-law solutions

$$a \propto t^{\frac{2n}{k^2}}, \qquad (22)$$

which inflate at all times when $k^2 < 2n$. Thus, the fields cooperate to make inflation more likely in the so-called assisted inflation [8]. The new solution (22) can be obtained from the seed solution (19) by using the transformation (12)-(14), generated by $\bar{\rho}(\bar{\phi}) = n^2 \rho(\phi)$. Here $\bar{\rho}$ and are the energy densities corresponding to the fields $\bar{\phi}$ (n-fields configuration) and ϕ (one-field configuration) respectively. Then, the set equations (17)-(18) and (20)-(21) become into the set equations (2)-(3) and (4)-(5) respectively. In this identification the energy-momentum tensor of the scalar fields and and have been written in the perfect fluid form

$$\rho(\phi) = \frac{1}{2}\,\dot{\phi}^2 + V\,,\;\; \bar{\rho}(\bar{\phi}) = \frac{1}{2}\,\dot{\bar{\phi}}^2 + \bar{V}, \qquad (23)$$

$$p(\phi) = \frac{1}{2}\,\dot{\phi}^2 - V\,,\;\; \bar{p}(\bar{\phi}) = \frac{1}{2}\,\dot{\bar{\phi}}^2 - \bar{V}, \qquad (24)$$

Then, the symmetry transformation (12)-(14) induces the following transformation rules for the scalar field and the potential

$$\dot{\bar{\phi}}^2 = n\,\dot{\phi}^2, \qquad (25)$$

$$\bar{V} = n^2\left(\frac{1}{2}\,\dot{\phi}^2 + V\right) - \frac{n}{2}\,\dot{\phi}^2\,. \qquad (26)$$

Actually, with $n > 1$, the transformation $\bar{\rho} = n^2\rho$ gives cooperative effects in the (n-fields configuration), $\bar{H} = n H$ increases the expansion of the universe and $\bar{a} = a^n$ relates (19) and (22) while the pressure of the transformed configuration $\bar{p} = -n^2 \rho + n \dot{\phi}^2$ becomes negative for increasing n. Hence, the symmetry transformation (12)-(15)) relates a scalar field configuration characterized by ρ, p and H with another field configuration characterized by $\bar{\rho}$, $\bar{p}$ and $\bar{H}$. In the last one the accelerated expansion is more likely for any potential whenever be $n > 1$. When the potential has the exponential form and n represents the number of scalar fields we get the usual assisted inflation. Here, it is obtained by adding energy density into the Friedmann equation and simultaneously imposing the form invariance of the EKG equations. We complete this section noting that the transformation rule of the remaining parameters g and q are given by Eqs. (16). They show that an expanding universe with a positive deceleration parameter transforms into an accelerated one by taking n large enough. This accelerated expansion scenario is corroborated by the present observational data.

IV. Duality and phantom cosmologies

Now we will consider two cosmological models, each one with a perfect fluid having equation of state $p = (\gamma - 1)\rho$ and $\bar{p} = (\bar{\gamma} - 1)\bar{\rho}$ respectively [19]. Here and throughout γ and $\bar{\gamma}$ stand for the constant barotropic indexes of the seed and transformed fluid respectively. Thus, the respective expansion rates become

$$H = \frac{2}{3\gamma t}, \qquad \bar{H} = \frac{2}{3\bar{\gamma} t} \qquad (27)$$

and the scale factors are the power-law solutions

$$a(t) = (\pm t)^{\frac{2}{3\gamma}}, \qquad \bar{a}(t) = (\pm t)^{\frac{2}{3\bar{\gamma}}}. \qquad (28)$$

From Eq. (27), we have expanding solutions for $\gamma t > 0$, $\bar{\gamma} t > 0$ and contracting ones for $t < 0$, $t < 0$. In order to be able to carry out the following discussion for a positive , we will take the plus sign for the $t > 0$ branch of the solution and the minus sign for the $t < 0$ one. The con-

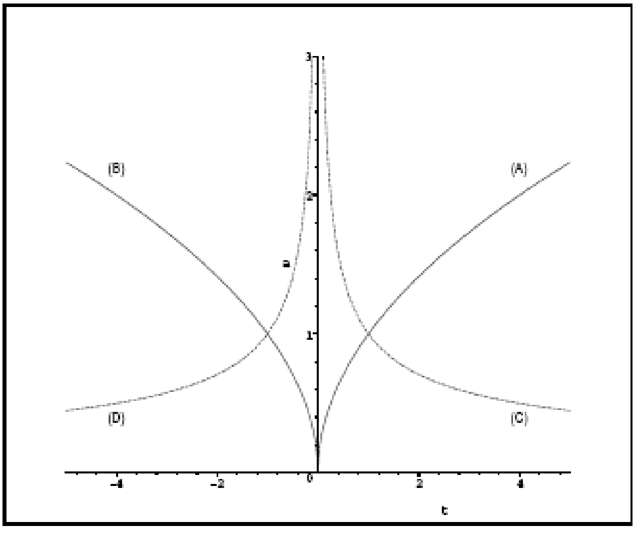

FIG. 1: The four branches of the scale factor Eq. (28). The continuous line, branch A, is a plot of $a(t) = (\pm t)^{2/3\gamma}$, with $\gamma > 0$ for a after-big bang cosmology ($t > 0$) and its pre-big bang continuation ($t < 0$), branch B. The dashed line represents a new solution obtained by applying to the latter a generalized duality transformation $a \to a^{|n|}$. Its $t > 0$ region, branch C, represents a contracting universes, while the $t < 0$ region, branch D, represents a phantom cosmology reaching a big-rip at the finite time $t = 0$.

nection between these two time regions is made throughout a time reversal symmetry and the two branches will be mirror images of each other. Since this transformation does not change the Einstein equation (2)-(3), there exists a form-invariance transformation between them and it is given by Eqs. (12), (14) and (16). Hence, one arrives at

$$|t|^{2/3\bar{\gamma}} = |t|^{2n/3\gamma}, \qquad \bar{\gamma} = \frac{\gamma}{n}. \qquad (29)$$

Since in the $n < 0$ case one has $a \to a^{-|n|}$ the transformation may be viewed as an extension of the well-known $a \to a^{-1}$ duality given in [18], and which was profusely studied in the pre-big bang scenario [20]. Now a solution of the form of (28) with $\gamma > 0$ describes for $t > 0$ a standard universe that expands ever after a big-bang at $t = 0$, but the $t < 0$ branch describes a contracting solution for which $t = 0$ represents a big crunch, (see Fig. 1). Now, when we apply the transformation we obtain a new solution with $\bar{\gamma} < 0$ which contracts in its $t > 0$ branch, but expands in the $t < 0$ one. Precisely, the latter is a phantom universe because it is a expanding solution that violates the weak energy condition. Moreover, it reaches a big rip at finite time, which is a feature inherent to many phantom universes but not to all of them. Thus, the phantom universe is dual to the universe described by the $t < 0$ branch of the seed solution.

A phantom universe contains a peculiar fluid with an equation of state, such that $\dot{\rho} > 0$. It is equivalent to the violation of the weak energy condition $(\rho + p) < 0$, meaning that $H > 0$ and $\dot{H} > 0$. Hence, one can infer two cases, it has an asymptote $\rho \to \Lambda$ when $t \to \infty$ or grows without limit. In the first case the scale factor tends to the de Sitter solution . But in the other case, if it is assumed that the energy density has the asymptotic behavior with $k > 0$, the asymptotic expanding solution of the Friedmann equation reads

$$a^{-} \to \left[\frac{2}{\eta\sqrt{\rho_0}\,(t_0 - t)}\right]^{\frac{2}{k}}, \quad t < t_0 . \qquad (30)$$

The expanding solution a^{-} is defined for $t < t_0$ and displays a big rip at $t = t_0$ because the scale factor diverges at the finite time t_0 and a future singularity occurs. Summarizing, the solution arises by phantomization of the solution $1/a^{-}$ which ends with a big crunch at $t = t_0$. In terms of form-invariance transformations the phantomization is originated by the $\frac{1}{a^{-}} \to a^{-}$ duality existing between those two solutions to the Einstein equations. In other words, for n = –1 that transformation becomes

$$\rho \to \bar{\rho} = \rho, \quad p + \rho \to \bar{p} + \bar{\rho} = -(p + \rho), \quad (31)$$

$$H \to \bar{H} = -H, \quad a \to \bar{a} = 1/a \qquad (32)$$

It transforms a contracting scale factor, $H < 0$, satisfying the WEC, $\rho + p > 0$ into an expanding one, $\bar{H} > 0$, which violates the WEC, so that $\bar{\rho} + \bar{p} < 0$. The latter, dubbed the dual transformation, is crucial because of its applicability as a method of transforming a conventional cosmological model into a phantom one.

V. Conclusions

We have found a symmetry transformation (6)-(8) under which, the EKG equations in FRW cosmology preserve their form. This symmetry group relates the expansion rate (geometrical variable) with the energy density and pressure of the perfect fluid (source variables). Hence, the cooperative effects of adding energy density into the Friedmann equation give rise to inflation. On the contrary, less energy density in the Friedmann equation tends to hinder the inflationary process. Therefore, it is possible to relate a non-accelerated scenario with an inflationary scenario by a symmetry transformation. We have studied the effects of the appearance of more than one scalar field in FRW space-time and have shown that assisted inflation, mainly developed with exponential potential and power-law solutions, can be seen as an application of a symmetry transformation between the configurations of one-scalar field and n-scalar fields.

Exploiting form-invariance transformations we have given a general prescription for generating phantom cosmologies which unifies those obtained from the EKG with both sign of the kinetic term. There, we have investigated linear transformations that multiply the Hubble factor by a negative constant factor n, so that contracting solutions will be transformed into expanding ones and vice versa. The transformation in the scale factor will be a generalized version of the $a \to a^{-1}$ duality investigated in [18] and appearing in the pre-big bang scenario, specifically our transformation will be $a \to a^{-|n|}$. Let us assume $n = -1$ and that the seed solution is the typical contracting (expanding) solution in the $t < 0$ ($t > 0$) region ending (beginning) in a singularity at $t = 0$ (see Fig. 1). Given that $a \to a^{-1}$, the $t < 0$ branch of the transformed solution will have an expanding character and it will reach a big rip at $t = 0$, thus representing a phantom cosmology. The $t > 0$ branches of the transformed and seed solutions are the standard duals of each other.

To sum up, we show how to obtain new solutions from existing ones using form invariance symmetries, relate non-accelerated and accelerated scenarios and vice versa, transform conventional scalar field cosmologies into phantom and view the duality as an application of form-invariance transformations. We think it is very interesting to study this kind of symmetry transformations and their associated symmetry group.

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